# Week 1

## Lecture 1: Piecewise

Piecewise functions:

## Lecture 2: Sets and Contradictions

Every set is a subset of itself. Every set has an empty set as a subset, too.

P = there is no largest number.

Assume not P = there is a largest number M.

M + 1 > M => M + 1 is larger.

Not P = False

Hence P = True

***CONTRADICTION***

HW:

P: There do not exist integers p, q such that 3p – 6q = 1

Not P: p and q exist

3p = 1 + 6q

3p = 1 + 2\*3q

3p / 3 = p (rem. 0)

(1 + 2\*3q) / 3 = k (rem. 1)

0 =/= 1

3p =/= 1 + 6q

Not P = false

P = true

## Lecture 3: Function Types

A polynomial’s degree cannot be less than 0.

Power functions:

* a = n – polynomial
* a = 1/n – root function
* a = -1 – reciprocal function

Power functions:

Transformations:

* Translations of functions: shifting the graph up/down/left/right
* Stretching: stretching or shrinking
* Reflection: -f reflects about x, -x reflects about y

Composite functions: =

## Lecture 4: Statements

One-to-one = injective functions: only one x value for any given y (always increasing or decreasing without vertical asymptotes)

* whenever

Negation of “for all” is “exists” and vice versa

# Week 2

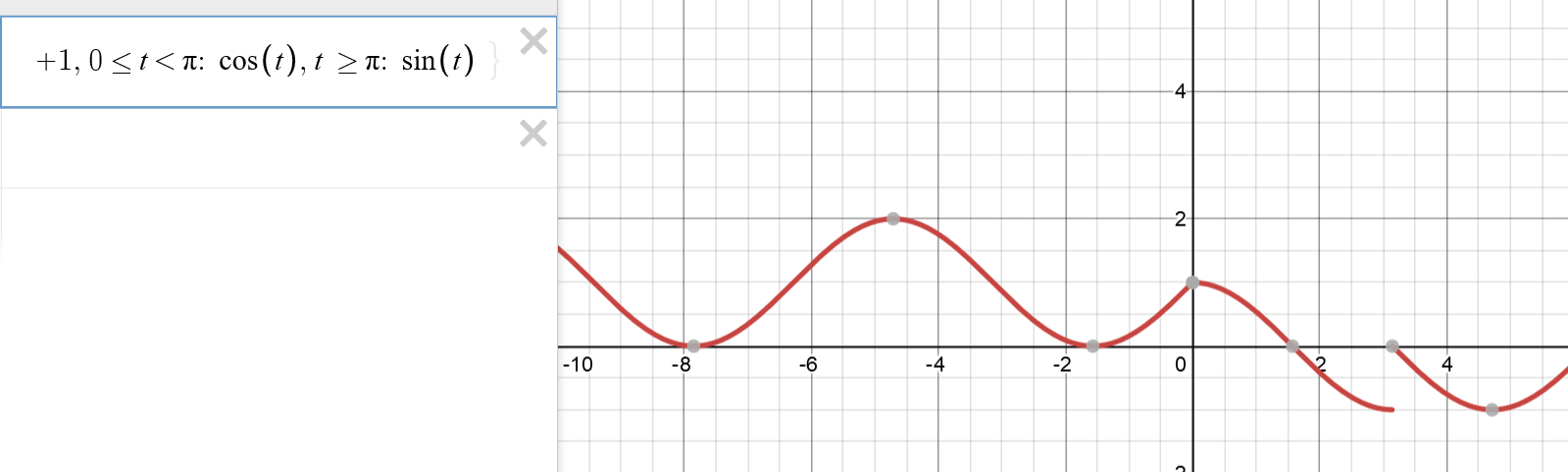
## Lecture 5: Limits and Tangent Lines

Tangent line – line that touches the curve and has the same direction as the curve at the point of contact

Secant line – line that intersects the curve more than once

Slope of the tangent line – limit of the slopes of the secant lines as neighbour points move closer to tangent point

One-sided limits:



Limit doesn’t exist

## Lecture 6: Number Systems

Real numbers are all numbers on the number line.

### Decimals

Geometric series:

* If -1<x<1, then
  + And

HW: What is the rational form of 0.22222…?

## Lecture 7: Limit Laws and Squeeze Theorem

Sum law:

Constant multiple law:

Product law, root law, difference law, power law – likewise

Squeeze theorem:

* If
* And
* Then

A function *f* is **continuous** at a number *a* if

Otherwise, it’s **discontinuous**

## Lecture 8: Decimal Periodicity, n­th Roots

Decimal expansion is **periodic** if

# Week 3

## Lecture 9: The Derivative (First Principles)

Tangent line: the line through a point with slope

Derivative at a:

## Lecture 10: Inequalities

The basic set of rules. **Everything has to be proved by these rules**.

1. If , x>0, x<0

## Lecture 11: Differentiability

Function is differentiable at *a* if f’(a) exists.

If *f* is differentiable at *a*, then *f* is continuous at *a*.

A function is not differentiable at *a* if:

* The function is not continuous, or
* There is a sharp corner: the gradient is discontinuous (proven by left and right limits)
* The function has a vertical tangent line

## Lecture 12: Modulus/Absolute Value

Modulus: x if x > 0 and -x if x < 0

* Geometric meaning: distance from origin 0 to point

# Week 4

## Lecture 13: Derivative Formulae

Leibniz notation:

**Reminder: don’t try to use the product rule for multiples of polynomials**

## Lecture 14: Complex Numbers

z = a + bi

a = Re(z) (real part of z)  
b = Im(z) (imaginary part of z)

The complex conjugate is

Every polynomial with n degrees has n complex roots (that’s including real numbers).

The roots are complex if

Principal argument:

## Lecture 15: Additional Formulae

## Lecture 16: Complex Roots

Nth roots of unity: complex numbers that satisfy the equation

All nth roots of unity:

# Week 5

## Lecture 17: Chain Rule

## Lecture 18: Polyn. Equations, Fund. Theorem of Algebra

Degree 1: linear equation  
Degree 2: quadratic equation  
Degree 3: cubic (**no need to know algorithm**)  
Degree 4: quartic  
Degree 5: quantic

A complex number is a root of

The Fundamental Theorem of Algebra:

* *Every polynomial equation of degree at least 1 has a root in C.* Hence:
* *For , every polynomial equation of degree n factorises as a product of linear polynomials and has exactly n roots in C (counting repeated roots).*

If all coefficients are real, the complex conjugate of root is also a root.

* *Every real polynomial factorises as a product of real linear and real quadratic polynomials. Moreover, the non-real roots always come in complex conjugate pairs.*

## Lecture 19: Chain Rule 2

It’s just practice

## Lecture 20: Polynomial Equations 2

N polynomial:

# Week 6

## Lecture 21: Implicit Differentiation

Given a function defined implicitly, differentiate both sides with respect to x.

## Lecture 22: Induction

The principle of mathematical induction:

Suppose that for each positive integer n there’s a statement P(n). If, for an integer *k*,

* The base case P(k) is true
* For all , if P(n) is true, then P(n+1) is also true;

then P(n) is true for all integers .

**Do not start with P(n+1).**

## Lecture 23: Taylor Series and Maclaurin Series

Taylor series:

Maclaurin series (a=0):

*n*th-degree Taylor polynomial of *f* at *a*:

## Lecture 24: Sigma Notation, Prime Factorisation

Example 8.8 Liebeck: reading

Principle of Strong Mathematical Induction is proved by Principle of Mathematical Induction.

# Week 7

## Lecture 25: Inverse Functions

If *f* has domain A, range B, has domain B, range A

Inverse function reflects about

Trigonometric:

* Sin:

If *f* is a one-to-one differentiable function with inverse function and , then

## Lecture 26: Euler’s Formula

Induction:

* Recurrence relation: equation that recursively defines a sequence. Once 1 or more initial terms are given, each further term of the sequence is defined as a **function of preceding terms**.
  + Logistic map: where
  + Fibonacci sequence: where

**Polyhedron**: a solid whose surface consists of a number of faces (polygons), such that any side of a face lies on exactly one other face.

* A polyhedron is called convex if, whenever one connects 2 points on the surface together by a straight line, that line is completely contained inside the polyhedron.
  + The Platonic solids are all convex.

**Planar graph**: figure in a plane consisting of a collection of points, and some points are connected with lines (not crossing).

* Planar graph is **connected** if one can get to all points from a point while following the lines
* It is **planar**if no lines cross

Let P(n) be: Every connected plane graph with *n* edges satisfies

* P(1) is with 2 vertices, 1 edge, 0 faces. True.
* Consider graph G with (n+1) edges:
* Strategy is to remove a selected edge so that G’ is still connected, then use P(n)
  + Case 1: If G has at least 1 face, removing an edge from the face leaves G’ still connected. Then , which is the same as
  + Case 2: G has an end vertex if it has no faces. Then G’ has v-1 vertices, n edges and 0 faces, then , which is the same as .
* If looking through a side of a polyhedron, we get a planar graph with v vertices, e edges and f-1 faces. Since , for polyhedrons we get .

## Lecture 27: Hyperbolic Functions

Similarity to trig:

## Lecture 28: Regular and Platonic Solids

Polygon is regular if all its sides are equal length and all its internal angles are equal.

Platonic Solids:

* Tetrahedron: 4 triangles
* Cube
* Octahedron: 8 triangles
* Dodecahedron

Regular polyhedron:

* V
* E
* F
* n: number of edges on the faces
* r: the number of edges each vertex belongs to

### Why only 5 regular polyhedrons:

Lemma: a subsidiary result used to prove a theorem.

Lemma 1:

* K is the number of pairs (e, f), where e is edge and f is face to which the edge is connected to
* Since each edge connects to two faces,
* Furthermore, each face has n distinct edges, so
* Therefore,

Lemma 2:

* L is the number of pairs (v, e)
* Every vertex is at the end of 2 distinct edges, so
* Each vertex meets r distinct edges, so
* Therefore,

Lemma 3:

* From lemma 1,

Lemma 4: If P is a regular convex polyhedron, then n = 3 or r = 3

* A polygon must have at least 3 sides (to be 3D), so
* At least 3 edges must meet any vertex, so
* Using lemma 3, RHS is larger than ½, so LHS must be, too
* Then and
* If both are 4, LHS is ½, which is not larger than ½
* If both are 5, LHS is smaller
* If one is 5 and the other is 4, LHS is 9/20, which is smaller
* Therefore, or

Then the theorem follows:

* Fix n = 3 (Lemma 4), RHS is 1/6 or greater
* For RHS to be greater, r = 3, r = 4, or r = 5 (also Lemma 4)
* Then there are 3 regular polyhedra
* Do the same for r = 3, then there are 2 more

# Week 8

## Lecture 29: Applications of differentiation

Absolute maximum: value of *f* on *D* if   
Absolute minimum: same, but less than

Local maximum: value of *f* if when x is near c  
Local minimum: same, but less than

Critical numbers: point *c* for which or does not exist

If:

* *f* is continuous on the closed interval [a,b]
* *f* is differentiable on the open interval (a,b)

then there is a number such that

## Lecture 30: The Integers and Euclidean Algorithm

Integers: whole numbers (positive, negative, or 0)

Let . a is a factor of b if b = ac for some integer c. a|b

Let a be a positive integer. Then for any , there are integers q, r such that

Let , and suppose d|a and d|b. Then for m,n in Z.

Euclidean Algorithm:

* Suppose , then divide a into b to get a quotient and remainder
* Divide into a, getting a quotient and remainder

If a,b in Z and d=hcf(a,b), then there are integers s and t such that

Two integers are called *coprime* if their highest common factor (hcf) is 1.

## Lecture 31: Local Minima and Maxima, Concavity

Local minimum: derivative changes from negative to positive as x increases  
Local maximum: derivative changes from positive to negative as x increases

Inflexion point: concavity changes at that point

## Lecture 32: Prime Factorisation, Fundamental Theorem of Arithmetic

Let *n* be an integer with :

* Then n is equal to a product of prime numbers

Where *p*s are primes and

* This prime factorisation of *n* is unique

Prime factorisation can be written as powers of unique primes.

Proof prime factorisation is unique:

* Assume it’s not:
* When cancelling primes from both sides, we get
* Since r1 divides left-hand side, it must divide right-hand side
* Therefore, r(

If

Then:

# Week 9

## Lecture 33: Vectors I

Vector: quantity that has magnitude and direction.

Two vectors are equivalent if they have the same length and direction (regardless of position)

Components of vectors: the coordinates of the end point of starting point is at origin.

Scalar multiple: multiplying vector by a scalar

Unit vector: vector with length 1 in the same direction

Standard basis vectors:

## Lecture 34: Prime Factorisation

Let *n*  be a positive integer. Then is rational if and only if *n* is a perfect square; that is, is rational if and only if for some integer *m*.

* The reverse implication is clear: Suppose for some positive integer *m*. Then is rational
* Forward implication: Suppose is rational, then , where r,s in Z. Then . Since all powers of r squared (RHS) prime factors are even, and s squared prime powers are even, n prime factors also have to be even. Then

Let *a* and *b* be coprime positive integers. Then

1. If *ab* is a square, then both *a* and *b* are squares
2. If *ab* is an nth power for some n in N, then both *a* and *b* are also nth powers.

A Diophantine equation is a polynomial equation in two or more variables in which only integer solutions are considered.

* For example, ,
* Can an even square exceed a cube by 1?
* Liebeck Chapter 11, Question 8c:

## Lecture 35: Vectors: Dot Product, Angle, Projections

**Dot product** = **scalar product** (sometimes **inner product**)

Angle between vectors:

The direction angles are angles that **a**makes with the positive x-,y-,z-axes, respectively.

* …
* …

Vector projection: a vector projected on a line/vector from another vector

Scalar projection: signed magnitude of the vector projection

Scalar projection of **b** onto **a**:

Vector projection of **b** onto **a**:

## Lecture 36: Prime Numbers

Gauss conjectures that estimates the number of primes below any value *n*.

Prime Number Theorem:

* Proved by Hadamard and Vallee-Poussin independently
* For a positive integer *n*, let pi(n) be the number of primes up to *n*. Then the ratio of pi(n) and tends to 1 as *n* tends to infinity.

The Goldbach conjecture:

* Every even integer greater than 2 is the sum of two primes

The Weak Goldbach conjecture:

* Every odd integer greater than 5 is the sum of three prime numbers.

Congruence of Integers:

* Let *m* be a positive integer. For *a,b* in Zifm|(b-a), we write , and say *a is congruent to b modulo m*.

Every integer is congruent to exactly one of the numbers from 0 to (m-1) modulo m.

* Let x in Z. There are integers q,r such that  
   with
* This implies that so m|(x-r), hence

# Week 10

## Lecture 37: Vectors: Cross Product

Cross product = vector product (because product is a vector)

## Lecture 38: Congruence of Integers

Suppose (mod m) and (mod m). Then

* (mod m)
* (mod m)

If (mod m) and *n* is a positive integer, then

## Lecture 39: Vectors: Scalar Triple Product

Two non-zero vectors are parallel if and only if their cross product is 0.

The cross product of two vectors is orthogonal to both

Scalar triple product:

* If the scalar triple product is 0, all three vectors are **coplanar** (lie in the same plane)
* Volume of the parallelepiped is the magnitude of the scalar vector product

Vector triple product:

## Lecture 40: Congruence Equations

If *a* and *m* are coprime integers and *x,y, in Z* such that (mod m), then (mod m).

* Since *a* and *m* are coprime,
* Hence, conclusion.

Linear congruence equation:

The linear congruence equation has a solution if and only if hcf(a, m)|b.

Let and define two binary operations

* Sum
  + (mod m)
* Product
  + (mod m)

For a prime *p* and an integer *a* where *p* does not divide *a*, then

# Week 11

## Lecture 41: Equations of Lines and Planes

Vector equation:

where ***r0*** is the position vector of a point on the line, and ***v*** is a vector parallel to the line.

Parametric equations of a line:

Symmetric equations of a line:

## Lecture 42: Congruence: FLT and Roots

### Fermat’s Little Theorem:

* Let *p* be a prime number, and let *a* be an integer that is not divisible by *p*. Then

Primality testing with FLT:

* Testing for 943
* Assume 943 is prime
* Check if (mod 943)
* (mod 943)
* Thus, 943 is not a prime

If *p* and *q* are distinct primes, and *a* is an integer not divisible by *p* or *q*, then

### Finding *kth* roots modulo *m*

Consider the equation

The solutions *x* are called kth roots of b modulo m. That is, in

*p* being a prime, and *k* a positive integer coprime to p-1. Then

1. there is a positive integer s such that
2. for any integer b not divisible by p, the unique solution to is

To find positive s, get integer j such that , then

## Lecture 43: Equations of Planes

Vector equation:

where **n** is a normal to the plane, , **r0** is position vector to a point on the plane.

Scalar equation:

where and

The angle between planes is the angle between their normal vectors.

Line of intersection for two planes:

where **r0** is the position vector from a point on both planes, and **v** is an orthogonal vector to both normal vectors.

## Lecture 44: Congruence: Secret Codes

RSA encryption:

* For prime numbers *p* and *q*,
* Find a large number *e* that is coprime to *M*
* Make *N* and *e* public. This is the **public key**
* *p* and *q* are secret
* Convert text to alphabetic codes
* Divide the code into segments, each *n* digits, segment value: *x*
* mod N

RSA decryption:

# Term 2

# Week 12

Peter Stewart

## Lecture 45: Systems of Linear Equations, Matrices

### SoLE

Engineering problems are often solved using computational methods (e.g. finite element method), for which efficiency is assured by solving systems of linear equations.

Linear equation:

System of linear equations – finite set of linear equations, and a solution to a SoLE is a vector that is simultaneously a solution to all equations in the system.

Consistent SoLE – has at least 1 solution. Inconsistent otherwise.

### Matrices

Matrix – rectangular array of numbers called the entries or elements of the matrix.

Entry for i-th row and j-th column.

is

Gather coefficients aij into a matrix A and gather xjs in a column vector x and bjs into a column vector b.

Augmented matrix:

[A|**b**] = [a… | b…]

Matrix in row echelon form:

* any all-zero rows are at the bottom
* in each non-zero row, the first non-zero entry (the leading entry) is to the left of any leading entries below it

Reduced row echelon form

* row echelon form
* the leading entry in each non-zero row is 1
* each column having a **leading** 1 has 0 everywhere else

Elementary row operations (OREs) to reduce a matrix to row echelon form:

* interchange two rows
* multiply a row by a non-zero constant
* add a multiple of a row to another row

## Lecture 46: Gaussian Elimination

Solving SoLE with Gaussian elimination (**KNOW THIS BY HEART**):

* Write augmented matrix
* Use EROs to reduce the augmented matrix to reduced row-echelon form
* If the system is consistent, use back substitution to solve the equivalent system corresponding to the reduced row-echelon form matrix

A SoLE is homogeneous if and only if the constant term in each equation is 0

* The zero-vector will always be **a** solution to the SoLE

If a solution to a SoLE can be expressed with parameters, there are infinite solutions.

## Lecture 47: Spanning Sets

Linear combination of a vector:

Span of is the set of all possible combinations of the vectors in S:

A SoLE with augmented matrix [A|**b**] is consistent if and only if **b** is a linear combination of the columns of A

* Let
* If S is a spanning set, the system is consistent independent of **b**

The set S is a spanning set for Rn if Span(S)=Rn. The vectors in S span Rn.

Adding more vectors in a spanning set will create a new spanning set.

## Lecture 48: Linear Independence

For two “vectors” in one dimension (**:**

* all numbers in R can be expressed with either of them

A set of vectors in Rn is linearly independent if and only if the only solution to

is c1,c2,…,ck = 0.

Vectors **v**1, **v**2, …, **v**k in Rn are linearly dependent if and only if at least one of them can be expressed as a linear combination of the others.

Let **v**1, …, **v**m be (column) vectors in Rn and let

be the matrix with columns **v**1, …, **v**m. Then **v**1, …, **v**m are linearly dependent if and only if the homogeneous linear system with augmented matrix [A|**0**] has a non-trivial solution.

Linear dependence means having multiple solutions to the corresponding SoLE.

If m > n, then any set of m vectors in Rn is linearly dependent.

# Week 13

## Lecture 49: Matrix Operations

A diagonal matrix:

* square (n x n)
* off-diagonal elements are 0

Identity matrix:

* diagonal matrix
* only 1s in the diagonal

Matrix addition:

* requires same dimensions

Scalar multiplication and addition:

* m x n -> m x n

Subtraction:

* scalar multiplication by -1 and matrix addition

**Matrix multiplication**:

* If A is m x **n** and B is **n** x r (same inner dimensions), then C = AB is the m x r matrix with (i,j)th entry given by
* **cij is the dot product of the ith row of A with jth column of B**

Matrix powers:

* If A is n x n and k a positive integer
  + A0=In
  + A2=AA
  + …

**Transpose** of m x n matrix A is n x m matrix AT obtained by interchanging the rows and columns of A

* A square matrix is **symmetric** if AT = A

A + B = B + A. Proof:

* Define both matrices with m x n elements
* Show result of LHS
* Show result of RHS
* Compare

Matrix multiplication rules:

* A(BC) = (AB)C
* A(B+C) = AB + AC
* (A+B)C = AC + BC
* k(AB) = (kA)B = A(kB)
* InA=A=AIn if A is m x n

## Lecture 50: Multiplication Principle, Binomial Coefficient and Theorem

Let P be a process which consists of *n* stages, and suppose for each the *r*th stage can be carried out in ar ways. Then P can be carried out in **a1a2…an different ways**.

* Example: lowercase letter 3 character string that starts with two characters and ends with a digit: then 26 x 26 x 10

Let S be a set consisting of *n* elements. Then the number of different **orderings** of the elements of S is

Where n is a natural number and r is a natural number less or equal to n, the **binomial coefficient** (“n choose r”) is

which is the number of r-element subset of {1,2,…,n}.

Binomial Theorem:

Let n be a positive integer, then

## Lecture 51: The Inverse of a Matrix

If A is an n x n matrix, an inverse of A is an n x n matrix A’ such that

* A’A=In
* AA’=In
* If A’ exists, A is invertible

If a matrix is invertible, then its inverse is unique. Inverse of A A-1.

If A is an invertible n x n matrix, then the SoLE given by has the unique solution given by

If A is 2 x 2 with elements a,b,c,d. A is invertible if ad-bc=/=0, in which case

* If det(A)=ad-bc=0, A is not invertible.
* If A is invertible, its inverse is invertible and equals A
* If A and B are invertible and of the same size, AB is invertible and

## Lecture 52: Ordered Selections

Ordered selections:

* Let S be a set. An ordered selection of n elements from S is a sequence
* with

Counting ordered selections:

* Set S with n elements
* 1) The number of ordered selections of r elements of S, allowing repetitions, is nr
* 2) The number of ordered selections of r distinct elements of S is

Partitions:

* Let n be a positive integer and let S be a set with n elements. A partition of S is a collection of non-empty subsets of S1,..,Sk such that each element of S lies in precisely one of the Si. The partition is ordered if we take into account the order in which the Si are labelled.

Multinomial Coefficients:

* Set S with n elements, ri non-negative integers satisfying
* The total number of ordered partitions of S into subsets S1,…,Sk of sizes r1,…,rk is denoted by
* and is called a multinomial coefficient.
* The formula is

Multinomial Theorem:

# Week 14

## Lecture 53: Techniques of Integration

For a general domain [a,b], we divide the domain into *n* equally sized strips of width

Approximating the area A under the graph of the continuous function *f* with the right Riemann sum Rn­:

Left Riemann sum Ln:

Lower Riemann sum ln:

where x\* is the argument for the minimum value in the interval .

Opposite for upper Riemann sum (un)

Definite integral:

* The area is a limit when number of rectangles *n* tends to infinity on any Riemann sum:
* where xi is any point in the interval (max, min, right, left, other)

Function is integrable if it is continuous on [a,b], or if it has only a finite number of jump discontinuities.

Properties of def integral:

* constant out
* sum of functions is sum of integrals
* subtraction same
* if , integral is greater or equal to 0
* if integral of f is geq to integral of g

## Lecture 54: The Algebra (Yoga) of Sets

Set S, subsets A and B. Their union:

* The union is the smallest subset of S that contains both A and B
* (inclusive “or”)

Set S, subsets A and B. Their intersection:

* The intersection is the largest subset which is contained in both A and B.
* (“and”)

A and B (subsets of S) are disjoint if

Distributivity law:

A and B (subsets of S): their difference is

The complement of A in S is

Cartesian product:

* Let A and B be sets
* The product A times B comes with projections
  + pA to A, pB to B

## Lecture 55: Techniques of Numerical Integration

Midpoint rule:

* where
* and (midpoint of end points of interval)

Trapezium rule:

Simpson’s Rule:

* higher-order approximation by using parabolae instead of straight lines
* Divide [a,b] into an even number of n subintervals of equal length delta x
* Consider consecutive pairs of intervals and approximate the integral over each pair

## Lecture 56: Sets and Counting

Finite and infinite sets:

* We call a set S **finite** if it has a finite number of elements
* for n elements. Cardinality of S is n
* We call a set T infinite if T is not finite

Inclusion-Exclusion principle:

* For finite A and B
* General principle:

where

Euler’s Totient Function

where

Powerset:

# Week 15

## Lecture 57: Fundamental Theorem of Calculus

A function is differentiable at a if there exists f’(a) and on an interval if it is differentiable at very point in that interval.

A function is continuous on an interval if it is continuous at every number in that interval

**Fundamental Theorem of Calculus:**

If f is a continuous on [a,b], then the function g defined by

is continuous on [a,b] and differentiable on (a,b), and

Function F is called antiderivative of f if F’(x)=f(x) for all x in an interval I. The most general antiderivative of f is

where C is an arbitrary constant.

If f is continuous on [a,b], then

where F is an antiderivative of f, meaning F’(x)=f(x)

Antiderivative =?= indefinite integral

means .

## Lecture 58: Equivalence Relations

A relation R on a set S is a subset If we often write to indicate they are related. (Note: not implied that t~s)

Conditions on relations:

* Reflexivity: R is **reflexive** if, for all ,
* Symmetry: R is **symmetric** if, for then if and only if
* Transitivity: R is **transitive** if, for then and implies

An **equivalence relation** is a relation which is reflexive, symmetric, and transitive.

**Equivalence class** of s is

The equivalence classes of R are the subsets cl(s) as s runs over the elements of S.

Relation to partitions:

* The equivalence classes with respect to ~ form a partition of S.
* Proof:
  + Since ~ is reflexive, s is in cl(s)
  + Suppose x is in cl(s) and x is in cl(t). Then s ~ x and t ~ x. By symmetry, x ~ t. By transitivity, s ~ t.
  + Hence, cl(s) = cl(t)
  + If cl(s1) and cl(s2) are distinct, their intersection is the empty set

## Lecture 59: Integrals of Special Functions

A function has to be one-to-one **and surjective** for there to be an inverse. The domain of a function is the range of its inverse and vice versa.

More general:

Inverse:

## Lecture 60: Relations+, Functions

Let be a partition of S. Defining s ~ t iff there exists an i in I with s, t in Si gives an equivalence relation ~ on S.

Antisymmetric relation: a ~ b and b ~ a iff a = b

Partially ordered set (poset): set S and R, usually written such that R is

* reflexive
* antisymmetric
* transitive

Liebeck definition:

* Let S and T be sets. A function f from S to T is a rule that assigns to each s in S a single element of T, denoted f(s).

If f(s) = t, we sometimes write

The arrow is good notation: a function transforms S to T in some wat; it is a process for producing an element of T from an element of S

Formal definition:

* A function f from S to T is a subset such that for each s in S there is a unique element f(s) = t.

If is a function, S is the **domain** (source) of *f* and T is the **codomain** (target) of *f*.

The image (range) of f is

# Week 16

## Lecture 61: Integration by Substitution, by Parts

Trig subs:

By parts:

## Lecture 62: Functions

Definitions:

* Function f (S to T) is **onto** (**surjective**) if f(S) = T
  + A surjective map is a surjection
  + A function f is **surjective** if and only if for every t in T there exists s in S with f(s) = t
* A function is one-to-one (**injective**) if for and implies that
* A function is **bijective** if it is both injective and surjective
  + A bijective map is a bijection

Cardinalities:

* Surjective:
* Injective:
  + Pigeonhole principle: if then there is no injection from S to T.
* Bijective:

## Lecture 63: Partial Functions, Improper Integrals

Cases:

* Q(x) (denominator) is a product of distinct linear factors
* Q(x) is a product of linear factors, some of which are repeated
  + exponents added to factors
* Q(x) contains irreducible quadratic factors, none of which are repeated
* Q(x) contains a repeated irreducible quadratic factor
  + The factor below has an exponential

Improper integrals: integrals where the interval is infinite or f has an infinite discontinuity

Improper integrals with infinity are **convergent** if the limit exists and **divergent** if it doesn’t

Comparison Theorem:

* If f is convergent, g is convergent
* If g is divergent, f is divergent

## Lecture 64: Composition, Inverse, Partitions

Composition:

* If f is S to T and g is T to U
* Composition of f with g is the function

(Notation: g o f is gf)

* If both are injective, gf is injective
* If both are surjective, gf is surjective
* If both are bijective, gf is bijective
* .
* If gf is injective, f is injective
* If gf is surjective, g is surjective

An inverse for if it exists, is a function such that

(1\_S: S to S)

* Since ff^(-1) is surjective, f is surjective
* Since f^(-1)f is injective, f is injective
* Hence, f is bijective

An injection is a subset of the target.

Partitions:

* …

Bonus meme: Yoneda Lemma (A thing is determined by what it maps to and what it’s mapped from)

# Week 17

## Lecture 65: Area and Volume Between Curves

Area between two curves f(x) and g(x) (both continuous, f geq g)

Volume:

* Let A(x) be a cross-sectional area (a continuous function) of a solid in a plane through x and perpendicular to the x axis, then

Solids of revolution: solids formed by an area revolved around an axis (x or y)

* Disk:
* Washer

Volume by cylindrical shells of revolution:

* Volume of solid by rotating around y-axis:

where 2pi x is the circumference, f(x) is the height, and dx is the width.

## Lecture 66: Permutations

Def of permutations:

* A permutation of a set S is a bijection

Composition:

* If f and g are permutations, fg and gf are, too.

Inverses:

* If f is a permutation, then there must be an inverse f-1

One permutation of {1,2,3,4,5} is

The set of permutations of {1,2,…,n} is Sn.

We can depict it as

(with dots in the middle)

Number of elements of Sn is n!.

Compositions:

* gf: first to f, then g
* Powers:
  + f0 = id.
  + for some defined fn
* Properties:
  + Closure: If f and g are in Sn, so is gf
  + Associativity: For any f,g,h in Sn,
  + Identity: The identity permutation id satisfies
  + Inverses: Every permutation has an inverse so that

Cycle notation:

* A permutation where can be written as a cycle (1 2).
* A permutation where can be written as two cycles (1)(2)
* Definition:
  + The cycle of a permutation is the permutation that sends
* Every permutation can be expressed as a product of disjoint cycles

## Lecture 67: Arc Length, Surface Area of Revolution

If f is continuous on [a,b], then the length of the curve is

The arc length function:

* if

Surface area of revolution:

* around x-axis:
* around y-axis:

## Lecture 68: Permutations (Cycles)

Length of a cycle: number of distinct elements. “n-cycles”

Every permutation can be expressed as a product of disjoint cycles

* Proof: TOO LONG
* This proves the statement.

Any cycle can be rotated without changing the permutation, as can the order of disjoint cycles be rearranged.

Multiplying cycles:

* (1 2)(2 3) = (1 2 3)
* (1 2 3)(3 2 5) = (1 2 5)(3)(4)

Cycle shape:

* Sequence of lengths of cycles in decreasing order
* Cycle shape for id is
* Cycle shape of (1 2 3)(5 6)(7 8 9) =

Order of a permutation:

* The least integer r>0 such that
* The least common multiple of the lengths of all disjoint cycles (**in cycle notation**)

Even and odd permutation:

* The parity of an r-cycle:
* Parity of a permutation:
  + Product of all parities of its cycles

# Week 18

## Lecture 69: Parametric Equations

\* - equations being functions of a parameter

If , (f(a),g(a)) is the initial point, (f(b),g(b)) is the terminal point.

Parametric curve: (x,y)=(f(t),g(t)) as t varies

Derivative:

Areas:

where and

Arc length differential:

## Lecture 70: Infinite Sets, Countable Sets

Cardinality of finite sets:

* |S|=|T| if and only if there exists a bijection between them
* Also: |S|=|T| if and only if there exists and injective maps f: S to T and an injective map g: S to T
  + Proof: pigeonhole principle in both directions

Cardinality of infinite sets:

* A and B have the same cardinality (|A|=|B|) if there is a bijection f: A to B
  + Equivalence relation
  + N and N x N are in bijection

Countability:

* A set A is countable (countably infinite) if it has the same cardinality as .
  + Every infinite subset of is countable
* If S is infinite and there is an injection f: S to , then S is countable.
* and cross … are countable
* (set of integers) is countable
  + There is a bijection
* (set of rational numbers) is countable
* is uncountable

## Lecture 71: Space Curves

Vector curves based on parametric equations:

A vector valued function is a function whose domain is a set of real numbers and whose range is a set of vectors.

Limit of r is the limit of each component.

Derivative:

Integral: (integral of curve is integral of each component)

Unit tangent:

Unit normal:

## Lecture 72: Uncountable Sets

is uncountable

* Proof: showing that is uncountable
  + Proof by contradiction (diagonal argument)

Computable reals: real numbers that can be computed to arbitrary precision

* The set of countable reals is countable (smaller than ), and it contains

Cardinality comparison:

* if there is an injective map . If there is no bijection, then
* Already proved that
* If and A isn’t empty, there is a surjection

Types of infinities:

Powerset of A, P(A), is the set of subsets of A.

* + Proof:
  + There is an injective function defined by
  + Thus
  + Proof by contradiction that there is no surjective function
  + Thus

Cantor’s Theorem:

* There is not greatest cardinality.
  + Proof: powerset of the “biggest” cardinality

Another Theorem:

* If M is a set of sets, there exists a set A that for all B in M
* Thus: There cannot be a set containing at least 1 set of each cardinality

Canter-Schröder-Bernstein Theorem:

* For infinite sets A and B, if and are injective,

Continuum hypothesis: the statement that there is a set with cardinality greater than N and smaller than R.

# Week 19

## Lecture 73: Differential Equations (Types)

A **differential equation** is an equation that contains an unknown function of one or more of its derivatives.

The **order** of a diff equation is the **order of the highest derivative** that occurs in the equation.

Applications of diff eq:

* Movement of pendulum
  + with the function is the angle to the vertical at time *t*
  + Formula for double pendulum also possible, as is for triple, etc

The general solution to a diff eq is a family of solutions that satisfy the eq, involving one or more arbitrary constants.

* The no. of arbitrary constants is equal to the order of the differential equation

Initial value problems (IVPs):

* There is a particular solution which satisfies an initial condition.

Boundary value problems (BVPs):

* The conditions are not all specified at the same point, e.g. thy may be specified at different ends of a domain.

Ordinary Differential Equations (**ODEs**) have 1 independent variable

Partial Differential Equations (PDEs) have more than 1 independent variable

A diff eq is known as **homogeneous** if all terms in the eq involve the dependent variable (usually *y*)or its derivatives. Otherwise, it is known as inhomogeneous.

A linear diff eq has the dependent variable and all its derivatives appear linearly. Otherwise

* Linearly: multiplied by function of x, but not y

Types of eq for finding solutions:

* Separable first order diff eqs (can be inhomogeneous and nonlinear) (9.3)
* Inhomogeneous linear first order diff eqs (9.5)
* Homogeneous linear second order diff eqs (17.1)
* Inhomogeneous linear second order diff eqs (17.2)

## Lecture 74: Groups (Characteristics)

A binary operation on a set S is a function

* Often written as

A binary operation always satisfies **closure** (operation on 2 elements of a set gives an element in the same set)

Associativity: For any we have

* A binary operation is **associative** if for all s,t,u in S

If is a set with an associative binary operation and is a subset which is closed under , then is associative.

A binary operation in S has an **identity** (or, is **unital**) if there exists an element such that for all

* e is the **identity element** (identity) or **unit**

with identity element e has inverses if for every there exists an with

## Lecture 75: Solutions to Diff Equations

Separable equation: first-order diff eq in which the expression for dy/dx can be factored as a function of x times a function of y. It can be written in the form

* The solution is:

## Lecture 76: Groups+ (Def, Cardinality)

A **group** is a set G with a binary operation such that

* the binary operation is associative
* there is a unit for the bin op
* every g in G has an inverse under the bin op

Group examples:

* Addition for

A group is **abelian** if for all g, h in G, we have

* addition is abelian, composition not

If |G| is **finite**, then we call G a **finite group**; otherwise, it’s an infinite group.

In a group G

* there is a **unique identity** element e in G
* each element has a **unique inverse**

# Week 20

## Lecture 77: Inhomogeneous 1st Order ODEs

A **first-order linear differential equation** is written in the form

First step: find integrating factor

Then the solution is

## Lecture 78: [REDACTED]

## Lecture 79: 2nd Order ODEs

Second-order linear diff eq:

* homogeneous if G(x)=0

If y1 and y2 are both solutions of the linear homogeneous eq and c1 and c2 are constants, then another solution (**the complementary function**) is:

A specific 2nd order (all functions of x are constants):

Solution:

Auxiliary equation:

Solution to this:

Cases:

* Two real distinct rs if
* Real and repeated r
* If the rs are complex ( and
  + The solution is real when the cs are real